Solutions to Midterm Exam

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Problem 1. Compute the radii of convergence of the following power series:

$$(a)\sum_{n=0}^{\infty} a^n z^{n^2} \qquad (b)\sum_{n=1}^{\infty} n^2 a^n z^{n^2-1},$$

where $a \in \mathbb{C}$ and $a \neq 0$.

Proof. (a)

$$\frac{1}{R} = \limsup_{n\to\infty} |a^n|^{1/n^2} = \limsup_{n\to\infty} |a|^{1/n} = |a|^0 = 1.$$

So $R = 1$.
(b) By Proposition 2.5 a) in Conway, this power series has the same radius of convergence as the one above. So $R = 1$ in this case too. $\square$

Problem 2. Let $S(z)$ be a Möbius transformation such that S maps lines in $\mathbb{C}$ to lines. Determine all such $S(z)$ that also have k fixed points in $\mathbb{C}$, where

$$(a) k \geq 2 \qquad (b) k = 1 \qquad (c) k = 0.$$

Proof. We know that Möbius transformations map circles in $\mathbb{C}_\infty$ to circles in $\mathbb{C}_\infty$. A line in $\mathbb{C}$ is the same as a circle in $\mathbb{C}_\infty$ that passes through ∞. So we want to look at all Möbius transformations S that send circles through ∞ to circles through ∞. We claim that this is same as saying that $S(\infty) = \infty$. Suppose not. Let S be a Möbius transformation that sends lines to lines, but with $S(\infty) = c \neq \infty$. Look at a very small ball B around ∞. This maps to a small ball around c. Choose B small enough such that the ball around c does not contain ∞. Then any circle inside B that passes through ∞ (i.e a line) maps to a circle inside the ball around c which cannot be a line as it does not pass through ∞.

Now note that if $S(z) = \frac{az+b}{cz+d}$, then $S(\infty) = \infty$ iff $c = 0$. But if $c = 0$, we know that $d \neq 0$ as $ad - bc \neq 0$. So just to make our notation simpler, we replace $\frac{a}{d}$ by a and $\frac{b}{d}$ by b. So we can write $S(z) = az + b$ with $a \neq 0$.
(a) If S has at least 2 fixed points in $\mathbb{C}$, then it has at least 3 fixed points in $\mathbb{C}_\infty$ (including ∞) and hence it is the identity.
(b) Suppose it has 1 fixed point. Then $S(w) = w$ for some $w \in \mathbb{C}$, i.e. $(a-1)w + b = 0$. Now this has a unique solution iff $a \neq 1$. So $S(z) = az + b$ has exactly one fixed point iff $a \neq 1$.
(c) S doesn't have any fixed point if $(a-1)z + b = 0$ has no solutions in $\mathbb{C}$. This happens iff $a = 1$ and $b \neq 0$, i.e. S is a translation. $\square$

Problem 3. Let $(X, d), (\Omega, p)$ be metric spaces, and $G \subset X, \Delta \subset \Omega$ open subsets. A map $f : G \to \Delta$ is called proper if $f^{-1}(K) \subset G$ is compact for every compact $K \in \Delta$. (The empty set is compact.) Suppose that $f : \overline{G} \to \overline{\Delta}$ is continuous and its restriction to G is a proper map $G \to \Delta$. Show that $f(\partial G) \subset \partial\Delta$.

Proof. If $\partial G = \emptyset$, we are done trivially. Suppose not. Let $x \in \partial G$. Suppose that $y = f(x)$ and $y \notin \partial\Delta$. Choose a sequence $\{x_n\}$ in G converging to x. Then $f(x_n) \to y$. Also, $f(x_n) \in \Delta$ for all n. Let us denote by S the set $\{y, f(x_n) : n \geq 1\}$. Then S is compact and $S \subset \Delta$. Then $(f|_G)^{-1}(S)$ is also compact as $f|_G$ is proper. But $\{x_n\} \subset (f|_G)^{-1}(S)$, but $x \notin (f|_G)^{-1}(S)$ which means that it cannot be compact. So $f(x) \in \partial\Delta$. This completes the proof. $\square$